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J. Differential Equations 225 (2006) 605–623
Journal of Differential Equations
www.elsevier.com/locate/jde

Convex curves moving translationally in the plane

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Received 23 December 2004; revised 1 March 2006

Available online 4 April 2006

Abstract

We show that a nontrivial homothetic self-similar solution can happen only when $F(k) = k^\alpha$ or $F(k) = -k^{-\alpha}$. We also derive a parametric representation of a translational self-similar solution. A translational self-similar solution may have self-intersections but cannot be a simple closed curve for any $F(k)$.

MSC: 35K15; 35K55

Keywords: Curvature flow; Homothetic self-similar solution; Translational self-similar solution

1. Introduction

Let $\gamma_t, t \in [0, T)$, be a family of convex[1] plane curves given by smooth immersions $X_t = X(\cdot, t): I \to \mathbb{R}^2$, where I is either S^1 or some open interval in $\mathbb{R}$. The curves γ_t are said to evolve under the $F(k)$ flow if we have

$$\frac{\partial X}{\partial t}(u, t) = F\big(k(u, t)\big) \cdot N(u, t) \quad \text{for all } (u, t) \in I \times [0, T) \tag{1.1}$$

where $k(\cdot, t)$ is the curvature of γ_t and $N(\cdot, t)$ is the unit normal vector of the curve γ_t. Here $F(k): \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ is a given, but arbitrary, smooth strictly increasing function of the

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[1] When a curve has self-intersections, convex always means "locally convex."

0022-0396/$ – see front matter
doi:10.1016/j.jde.2006.03.005

curvature. We use the convention that for convex plane curves $k > 0$ and N points into the convex region enclosed by γ_t in case it is closed and embedded.

We would like to ask some questions regarding some types of self-similar solutions to Eq. (1.1), more specific, homothetic self-similar solutions and translational self-similar solutions. A convex solution $X(u,t): I \times [0,T) \to \mathbb{R}^2$ of Eq. (1.1) is called *self-similar* if the shape of $\gamma_t = X(\cdot,t)$ is independent of time (up to a homothety). If there is a center point with respect to which $\gamma(\cdot,t)$ shrinks or expands, $\gamma(\cdot,t)$ is called *homothetic self-similar*. If $\gamma(\cdot,t)$ translates along a direction only, we call it *translational self-similar*. Finding these self-similar solutions is more or less an ODE problem.

Two important classes of Eq. (1.1) are the following homogeneous flows: $F(k) = k^\alpha$ or $F(k) = -1/k^\alpha$, where $\alpha > 0$ is a constant. Of which there is the famous curve shortening flow by Gage and Hamilton [15] and Grayson [14] when $F(k) = k$. One can see the references in Andrews [1] for a big literature on the study and development of these two flows. For the case $F(k) = k^\alpha$ or $F(k) = -1/k^\alpha$ with general $\alpha > 0$, convex homothetic self-similar solutions have been completely classified by Andrews [2,3] and Urbas [17,18]. For $\alpha = 1$ and γ_t is convex closed and immersed, classification was first done by Abresch and Langer [4]. For the special interesting case $F(k) = k^{1/3}$, see Andrews [3]. For $F(k) = k^\alpha$, $\alpha > 0$, these homothetic self-similar solutions undergo a contraction to a point with their shapes unchanged. Each such curve satisfies the relation $k^\alpha = \lambda\langle X, N_{\text{out}}\rangle$, where $\lambda > 0$ is a constant and N_{out} is outward unit normal. The solution of (1.1) with this initial data is, up to a reparametrization in space at each time, given by $\gamma_t = [1 - \lambda(1+\alpha)t]^{1/(1+\alpha)}\gamma_0$. Similar results hold for the case $F(k) = -1/k^\alpha$.

The main content of this paper is on translational self-similar solutions. For homothetic self-similar solutions we just point out that nontrivial (i.e., noncircular) homothetic self-similar solution to Eq. (1.1) can occur only when $F(k) = k^\alpha$ or $F(k) = -1/k^\alpha$ for some constant $\alpha > 0$, that is, only when the flow is homogeneous (so that any spacial dilation, when combined with a suitable rescaling of time, is a symmetry of the equation). This amounts to a brief computation. For other types of flow, nontrivial homothetic self-similar solution does not exist (unless $F(k) = k^\alpha$ or $F(k) = -1/k^\alpha$ over some interval $k \in I \subset \mathbb{R}_+$). The fact that nonhomogeneous flows cannot have (noncircular) homothetic self-similar solutions raises the question of what behavior should be expected near singularities in such cases. There is much territory to explore here and unfortunately they are not discussed in any detail in this paper.

For translational self-similar solutions γ_t to Eq. (1.1), we derive a parametric description of them (in terms of a pair of integrals giving the coordinates of the curve in the Gauss map parametrization). From it, we see that for any $F(k)$, if we adjust the translation speed of γ_t suitably, a translational self-similar solution to Eq. (1.1) can always be found. We also observe that a translational self-similar solution may have self-intersections, but it cannot be a simple closed convex curve. Most of the paper is devoted to examples of the resulting translational self-similar solutions for various flows, together with observations on the allowed speeds of the translational self-similar solutions and their domains of definition via the Gauss map. Some of the examples are quite interesting, though elementary.

The importance of these self-similar solutions is that they describe the asymptotic behavior of solutions. For example, in the most important case $F(k) = k$, Angenent [5] shows that if a convex closed immersed curve shrinks to a point, its asymptotic shape must be one of the contracting homothetic self-similar solutions classified in [4]. On the other hand, the asymptotic behavior of a "type-II" singularity looks like a "grim reaper," which is a translational self-similar solution. This "grim reaper" solution was also used by Hamilton [16] to give a different proof of Grayson's result in [14]. It is not known if similar results hold under the general case $F(k) = k^\alpha$, $\alpha > 0$.

There is also some interest in nonhomogeneous flows. For example, Chou and Wang [12] discussed the logarithmic Gauss curvature flow of a convex closed embedded hypersurface and Chow and Tsai [9,10] discussed the general nonhomogeneous flow of a convex closed embedded hypersurface. See also [1] for several types of nonhomogeneous curve flows in the plane. Again, even if there is an interesting nonhomogeneous flow there is no obvious argument to show that the limiting solution must be a translational self-similar solution. Please also see [6–8,11] for interesting results related to evolution of convex curves in the plane.

2. Homothetic self-similar solutions

Assume that $\gamma_t = X(u,t): I \times [0,T) \to \mathbb{R}^2$ is a convex homothetic sss (self-similar solution) to Eq. (1.1), then there is a point $q \in \mathbb{R}^2$ (center of homothety) such that

$$\gamma(N,t) - q = c(t) \cdot \big(\gamma(N,0) - q\big) \quad \text{for all } N \tag{2.1}$$

for some positive function $c(t)$ with $c(0) = 1$. Here $\gamma(N,0)$ is the point on the curve γ_0 with normal N and $\gamma(N,t)$ is the point on the curve γ_t with the same normal N. This is due to the fact that the normal vector field is unchanged under homotheties and translations. Geometrically (2.1) means that the image of γ_t comes from a dilation of γ_0. Without loss of generality, from now on we shall assume $q = 0$ and so

$$\gamma(N,t) = c(t)\gamma(N,0) \tag{2.2}$$

for all $N \in I$ describing the convex curve γ_0.

(2.1) suggests that if at each time t we perform a reparametrization in space of the solution $X(u,t)$ to keep its normal unchanged, then X will become a separable solution under the new variable. Geometrically this is the same as adding a tangential component to the flow. Thus we consider the modified flow

$$\frac{\partial X}{\partial \tau}(v,\tau) = F(k)N + fT, \quad (v,\tau) \in J \times [0,T) \tag{2.3}$$

where J is some open interval (or S^1) and f is to be chosen so that

$$\frac{\partial N}{\partial \tau}(v,\tau) = 0 \quad \text{for all } (v,\tau). \tag{2.4}$$

The modified flow (2.3) is geometrically the same as (1.1) but has the advantage that when a convex solution to Eq. (1.1) is homothetic self-similar, then it satisfies Eq. (2.3) in a *separable* way.

To determine f, let $w = |\partial X/\partial v|$ and so $\partial/\partial s = w^{-1}\partial/\partial v$, where s is the arc length parameter. Compute

$$\frac{\partial w}{\partial \tau} = w\left\langle \frac{\partial}{\partial s}\left(\frac{\partial X}{\partial \tau}\right), \frac{\partial X}{\partial s}\right\rangle = w\left\langle F(k)\frac{\partial N}{\partial s}, T\right\rangle + w\left\langle \frac{\partial f}{\partial s}T,\ T\right\rangle = -wkF(k) + w\frac{\partial f}{\partial s}.$$

Therefore

$$\frac{\partial}{\partial \tau}\frac{\partial}{\partial s} = \frac{1}{w}\frac{\partial}{\partial v}\left(\frac{\partial}{\partial \tau}\right) - \frac{1}{w}\left(-kF(k) + \frac{\partial f}{\partial s}\right)\frac{\partial}{\partial v} = \frac{\partial}{\partial s}\frac{\partial}{\partial \tau} + \left(kF(k) - \frac{\partial f}{\partial s}\right)\frac{\partial}{\partial s}. \tag{2.5}$$

The time derivative of T is then given by

$$\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial \tau}\frac{\partial X}{\partial s} = \frac{\partial}{\partial s}\bigl(F(k)N + fT\bigr) + \left(kF(k) - \frac{\partial f}{\partial s}\right)T = \left(\frac{\partial F(k)}{\partial s} + fk\right)N$$

and hence

$$\frac{\partial N}{\partial \tau} = -\left(\frac{\partial F(k)}{\partial s} + fk\right)N. \tag{2.6}$$

Hence if we choose f as

$$f = -\frac{1}{k}\frac{\partial F(k)}{\partial s}$$

we would have $\partial T/\partial \tau = \partial N/\partial \tau = 0$.

We conclude the following:

Lemma 2.1. *Under the modified flow*

$$\frac{\partial X}{\partial \tau}(v, \tau) = F(k)N - \left(\frac{1}{k}\frac{\partial F(k)}{\partial s}\right)T \tag{2.7}$$

we have

$$\frac{\partial T}{\partial \tau}(v, \tau) = \frac{\partial N}{\partial \tau}(v, \tau) = 0 \quad \textit{for all } (v, \tau). \tag{2.8}$$

Thus for fixed v, *the normal along the time curve* $X(v, \tau)$ *is the same for all* τ.

The commutation formula (2.5) now becomes

$$\frac{\partial}{\partial \tau}\frac{\partial}{\partial s} = \frac{\partial}{\partial s}\frac{\partial}{\partial \tau} + Q\frac{\partial}{\partial s}, \quad Q := kF(k) + \left(\frac{1}{k}\frac{\partial F(k)}{\partial s}\right)_s \tag{2.9}$$

and so under the modified flow, the evolution of the curvature k is given by the parabolic equation (since F is increasing)

$$\begin{aligned}\frac{\partial k}{\partial \tau} &= \frac{\partial}{\partial \tau}\left\langle \frac{\partial T}{\partial s}, N\right\rangle = \left\langle \frac{\partial}{\partial s}\frac{\partial T}{\partial \tau} + Q\frac{\partial T}{\partial s}, N\right\rangle = kQ \\ &= F(k)_{ss} - \frac{k_s}{k}F(k)_s + F(k)k^2 = \frac{F'(k)}{w^2}k_{vv} + \cdots. \end{aligned} \tag{2.10}$$

The modified flow (2.7) is geometrically the same as the original flow (1.1). For a homothetic sss X to (1.1), by (2.2) we have

$$X(v, \tau) = c(\tau) \cdot X(v, 0) \quad \text{for all } (v, \tau) \tag{2.11}$$

where $X(v,\tau)$ satisfies (2.7). Differentiating (2.11) with respect to τ gives

$$F(k)N - \left(\frac{1}{k}\frac{\partial F(k)}{\partial s}\right)T = c'(\tau)\cdot X(v,0) = \frac{c'(\tau)}{c(\tau)}\cdot X(v,\tau). \tag{2.12}$$

Taking normal and tangential components of (2.12) yields the following:

Lemma 2.2. *We have*

$$F\big(k(v,\tau)\big) = \frac{c'(\tau)}{c(\tau)}\cdot \big\langle X(v,\tau), N(v,\tau)\big\rangle \tag{2.13}$$

and

$$\left(\frac{1}{k}\frac{\partial F(k)}{\partial s}\right)(v,\tau) + \frac{c'(\tau)}{c(\tau)}\cdot \big\langle X(v,\tau), T(v,\tau)\big\rangle = 0 \tag{2.14}$$

for all (v,τ).

Using the relation $k(v,\tau) = k(v,0)/c(\tau)$ for a homothetic sss, we have

$$F\left(\frac{k(v,0)}{c(\tau)}\right) = c'(\tau)\cdot \big\langle X(v,0), N(v,0)\big\rangle \quad \text{for all } v \text{ and } \tau. \tag{2.15}$$

When F is a power-type function (homogeneous of certain degree), we can separate space and time in (2.15) to classify homothetic sss by solving an ordinary differential equation for $X(v,0)$. For nonhomogeneous function F, in general we cannot do so unless the homothetic sss is trivial, i.e., a circular solution (which means it is a family of shrinking circles or expanding circles centered at the origin $q=0$). More precisely, we have:

Lemma 2.3. *In* (2.15)*, unless both* $k(v,0)$ *and* $\langle X(v,0), N(v,0)\rangle$ *are constants, we must have* $F(z)=\lambda z^{\alpha}$ *or* $F(z) = -\lambda z^{-\alpha}$ *for some constants* $\lambda,\alpha>0$, *for all* z *belongs to some interval* $I\subset\mathbb{R}_+$. *That is*: *nontrivial homothetic sss occur only in the case* $F(z)=\lambda z^{\alpha}$ *or* $F(z)=-\lambda z^{-\alpha}$ *for all* $z\in\mathbb{R}_+$ *or for all* $z\in I\subset\mathbb{R}_+$. *Otherwise, the only homothetic sss are circles.*

Proof. Differentiate (2.15) with respect to τ to get

$$\frac{-c'(\tau)}{c(\tau)}F'(z)z = c''(\tau)\big\langle X(v,0), N(v,0)\big\rangle, \quad z = \frac{k(v,0)}{c(\tau)}.$$

Rewrite (2.15) as

$$\big\langle X(v,0), N(v,0)\big\rangle = \frac{F(z)}{c'(\tau)}$$

to get

$$\frac{F'(z)z}{F(z)} = -\frac{c''(\tau)c(\tau)}{c'(\tau)^2}, \quad \text{where } z = \frac{k(v,0)}{c(\tau)} > 0. \tag{2.16}$$

On the other hand, differentiate (2.15) with respect to v to get

$$F'(z)\frac{1}{c(\tau)}\cdot\frac{d}{dv}k(v,0)=c'(\tau)\cdot\frac{d}{dv}\big\langle X(v,0),N(v,0)\big\rangle \tag{2.17}$$

and so

$$F'(z)z\cdot\frac{d}{dv}k(v,0)=c'(\tau)k(v,0)\cdot\frac{d}{dv}\big\langle X(v,0),N(v,0)\big\rangle.$$

Dividing by $F(z)$, we get

$$\frac{F'(z)z}{F(z)}\cdot\frac{d}{dv}k(v,0)=\frac{k(v,0)}{\langle X(v,0),N(v,0)\rangle}\cdot\frac{d}{dv}\big\langle X(v,0),N(v,0)\big\rangle. \tag{2.18}$$

By (2.16) and (2.18), $F'(z)z/F(z)$ must be a constant independent of space and time. Hence

$$\frac{F'(z)z}{F(z)}=\alpha \quad \text{for some constant } \alpha\in\mathbb{R}$$

for all z in some interval $I\subset\mathbb{R}_+$. If $\alpha>0$, then since both $F'(z)$ and z are positive, we must have $F(z)>0$ for all $z\in I$. Integration gives $F(z)=\lambda z^{\alpha}$ where $\lambda>0$ is a constant. If $\alpha<0$, then we must have $F(z)<0$ for all $z\in I$ and integration gives $F(z)=-\lambda z^{-\beta}$ where $\lambda,\beta>0$ are two positive constants. The proof is done. □

3. Translational self-similar solutions

Now we assume that solution $X(u,t): I\times[0,T)\to\mathbb{R}^2$ to Eq. (1.1) is a translational sss. As the normal vector field is unchanged under translations, there exists a constant translation vector $V\in\mathbb{R}^2$ such that

$$\gamma(N,t)=\gamma(N,0)+tV \quad \text{for all } N. \tag{3.1}$$

Here $\gamma(N,0)$ is the point on the curve γ_0 with normal N and $\gamma(N,t)$ is the point on γ_t with the same normal N.

Let $X(v,\tau)$ be the corresponding solution of the modified flow (2.7) with initial data $\gamma_0=X(v,0)$. We now have

$$X(v,\tau)=X(v,0)+\tau V \quad \text{for all } (v,\tau). \tag{3.2}$$

Differentiating both sides of (3.2) with respect to τ gives

$$F(k)N-\left(\frac{1}{k}\frac{\partial F(k)}{\partial s}\right)T=V$$

and hence

$$\big\langle V,N(v,\tau)\big\rangle=F\big(k(v,\tau)\big)$$

which is the same as (note that $N(v,\tau)=N(v,0)$ and $k(v,\tau)=k(v,0)$ for a translational sss)

$$\bigl\langle V, N(v,0)\bigr\rangle = F\bigl(k(v,0)\bigr) \quad \text{for all } v. \tag{3.3}$$

We call (3.3) *the equation of translational sss*. Similarly, taking the tangential component, we find

$$\bigl\langle V, T(v,0)\bigr\rangle = -\left(\frac{1}{k}\frac{\partial F(k)}{\partial s}\right)(v,0) \quad \text{for all } v. \tag{3.4}$$

Note that (3.4) also comes from (3.3) by differentiating both sides with respect to s.

Differentiating (3.4) with respect to s again and by (3.3), we obtain the equation

$$Q := kF(k) + \left(\frac{1}{k}\frac{\partial F(k)}{\partial s}\right)_s = 0.$$

Thus on a translational sss, by (2.10) and (2.9), we have

$$\frac{\partial k}{\partial \tau}(v,\tau) = 0 \quad \text{and} \quad \frac{\partial}{\partial \tau}\frac{\partial}{\partial s} = \frac{\partial}{\partial s}\frac{\partial}{\partial \tau}.$$

Note that $X(v,\tau)$ and $X(v,0)$ have the same normal. If we parametrize the initial convex curve such that $X(v,0)$ has tangent vector $T(v,0)=(\cos v,\sin v)$ (i.e., we use tangent angle to parametrize γ_0; this is possible since γ_0 is convex), then the unit tangent at $X(v,\tau)$ is also given by $T(v,\tau)=(\cos v,\sin v)$ by (2.8). In doing so, we can interpret v as the tangent angle and have $N(v,\tau)=(-\sin v,\cos v)$ for all (v,τ).

By rotation, we may assume $V = A\cdot(0,1)$ for some constant $A>0$ where $A=|V|$ is the speed of translation. We first assume $A=1$ and would like to construct a curve with curvature k satisfying the equation

$$\bigl\langle V, N(v,0)\bigr\rangle = \cos v = F\bigl(k(v,0)\bigr) \quad \text{for all } v. \tag{3.5}$$

Assume the curve is given by $(x(v),y(v))$. By the formula

$$k = \frac{dv}{ds} \quad (v \text{ is tangent angle}), \qquad \frac{1}{k} = \frac{ds}{dv} = \sqrt{\bigl(x'(v)\bigr)^2 + \bigl(y'(v)\bigr)^2}$$

we have

$$x'(v) = \frac{\cos v}{k(v,0)}, \qquad y'(v) = \frac{\sin v}{k(v,0)}, \quad \text{where } F\bigl(k(v,0)\bigr) = \cos v \tag{3.6}$$

which yields the following *parametric form of a translational sss*

$$x(v) = \int_0^v \frac{\cos\theta}{k(\theta)}\,d\theta = \int_0^v \frac{\cos\theta}{F^{-1}(\cos\theta)}\,d\theta, \qquad y(v) = \int_0^v \frac{\sin\theta}{k(\theta)}\,d\theta = \int_0^v \frac{\sin\theta}{F^{-1}(\cos\theta)}\,d\theta, \tag{3.7}$$

where $k(\theta)=k(\theta,0)$.

Remark 1. In (3.7) we assume that the angle $v = 0$ is in the domain of $(x(v), y(v))$. We also have performed a translation in the curve so that $(x(0), y(0)) = (0, 0)$. The maximum curvature of the curve is $F^{-1}(1)$, occurring at $(0, 0)$. For other situation, one may have to integrate on a different interval.

There are some advantages for this parametric representation. For example, one can easily get the *grim reaper* $y = \log\sec x$ when $F(k) = k$ (the curve shortening flow) without solving a differential equation. We now have $F(k(v, 0)) = k(v, 0) = \cos v$. Since $k > 0$, we choose the domain of v to be $v \in (-\pi/2, \pi/2)$ and obtain $x(v) = v$ and

$$y(v) = \int_0^v \frac{\sin\theta}{\cos\theta}\, d\theta = \log\sec v, \quad v \in (-\pi/2, \pi/2).$$

One easily verify that for the graph given above the upward normal is $N = (-\sin v, \cos v)$ and so

$$k = \big\langle (0, 1), N \big\rangle = \cos v > 0 \quad \text{for all } v \in (-\pi/2, \pi/2). \tag{3.8}$$

It gives a translational sss moving in the direction $V = (0, 1)$.

Let $E \subset \mathbb{R}$ be the range of the strictly increasing function $F(z): (0, \infty) \to \mathbb{R}$. If $E \cap (-1, 1) = \emptyset$, then the flow (1.1) has no unit speed translational sss. For example, $F(k) = e^k$ or $F(k) = -e^{1/k}$. We also make the following simple observation:

Lemma 3.4. *Assume* $F^{-1}(\cos v)$ *is defined on the interval* $v \in (-\delta, \delta)$ *for some* $\delta > 0$*. Then* $\gamma = (x(v), y(v))$ *given by* (3.7) *is defined on* $v \in (-\delta, \delta)$ *such that* $x(v)$ *is an odd function of* v *and* $y(v)$ *is an even function. Moreover if* $\delta > 2\pi$*, then* $x(0) = 0$*,* $x(2\pi) < 0$ *and* $y(0) = y(2\pi) = 0$*,* $y(v) > 0$ *on* $v \in (0, 2\pi)$*.*

Proof. The first part is trivial. Assume $\delta > 2\pi$. Let $H(\sigma)$ be such that $H'(\sigma) = 1/F^{-1}(\sigma)$ for all $\sigma \in [-1, 1]$. Then

$$y(2\pi) = \int_0^{2\pi} \frac{\sin\theta}{F^{-1}(\cos\theta)}\, d\theta = \big[-H(\cos\theta)\big]_{\theta=0}^{\theta=2\pi} = 0$$

and if we look at the derivative of $y(v)$, we see that $y(v)$ is increasing on $(0, \pi)$, decreasing on $(\pi, 2\pi)$ and so $y(v) > 0$ on $(0, 2\pi)$. Also

$$x(2\pi) = \int_0^{\pi} \frac{\cos\theta}{F^{-1}(\cos\theta)}\, d\theta + \int_{\pi}^{2\pi} \frac{\cos\theta}{F^{-1}(\cos\theta)}\, d\theta = I + II$$

where

$$I = \int_0^{\pi} \frac{1}{F^{-1}(\cos\theta)}\, d(\sin\theta) = \int_0^{\pi} \sin^2\theta \cdot H''(\cos\theta)\, d\theta.$$

As $1/F^{-1}(\sigma)$ is a strictly decreasing function of σ, we have

$$H''(\sigma)=\frac{d}{d\sigma}\left(\frac{1}{F^{-1}(\sigma)}\right)\leqslant 0 \quad \text{for all } \sigma=\cos\theta,\ \theta\in[0,\pi]$$

which implies that $I\leqslant 0$. Similarly

$$II=\int_{\pi}^{2\pi}\sin^2\theta\cdot H''(\cos\theta)\,d\theta\leqslant 0.$$

Thus we have $x(2\pi)<0$. $\square$

Locally a translational sss can be represented as a graph $y=h(x)$ over some interval $x\in I$. One can derive the differential equation for $h(x)$ easily using the parametric form (3.7). On any interval J of v where $\cos v$ does not change sign (say $\cos v>0$) we have

$$\frac{dx}{dv}=\frac{\cos v}{F^{-1}(\cos v)}>0, \quad v\in J\subset(-\pi/2,\pi/2).$$

Hence $h'(x)=dy/dv\cdot dv/dx=\tan v$ and

$$h''(x)=\sec^2 v\,\frac{F^{-1}(\cos v)}{\cos v}=\bigl(1+\bigl(h'(x)\bigr)^2\bigr)^{3/2}F^{-1}\left(\frac{1}{\sqrt{1+(h'(x))^2}}\right), \quad x\in I, \tag{3.9}$$

which is the differential equation of the graph of a translational sss.

Another interesting observation is the following: a convex translational sss can have self-intersections, but it cannot be a convex simple closed curve.

Lemma 3.5. *There does not exist a convex closed embedded curve as a translational sss for any curvature function $F(k)$.*

Proof. For a convex closed curve to be a translational sss we must first assume that $F^{-1}(\cos v)$ is defined on an interval $v\in I$ with length at least 2π. This actually implies that $F^{-1}(\cos v)$ is defined on all $v\in\mathbb{R}$. We take $I=[0,2\pi]$. We recall the following result due to Gage and Hamilton [15]: *Let $k(\theta)>0$ be a positive smooth 2π-periodic function. Then $k(\theta)$ represents the curvature function of a smooth simple closed convex plane curve if and only if*

$$\int_0^{2\pi}\frac{\cos\theta}{k(\theta)}\,d\theta=\int_0^{2\pi}\frac{\sin\theta}{k(\theta)}\,d\theta=0. \tag{3.10}$$

Since we already know that $x(2\pi)=\int_0^{2\pi}\cos\theta/F^{-1}(\cos\theta)\,d\theta<0$, where $F^{-1}(\cos\theta)$ is the curvature of the translational sss, it is impossible to have a convex closed embedded curve as a translational sss. The proof is done. $\square$

Remark 2. However we can have a translational sss which is almost a closed curve when $v\in[0,2\pi]$. See Example 5 below.

3.1. Examples of unit-speed translational sss

We shall look at some examples below and plot their pictures. They are simple but interesting.

Example 1. Consider the logarithmic Gauss curvature flow as by Chou and Wang [12], where

$$F(k) = \log k, \quad F : \mathbb{R}_+ = (0, \infty) \to \mathbb{R}.$$

We have $k(v) = e^{\cos v} > 0$ and can allow the domain of v to be $v \in (-\infty, \infty)$. Now

$$x(v) = \int_0^v \frac{\cos\theta}{e^{\cos\theta}}\, d\theta, \qquad y(v) = \int_0^v \frac{\sin\theta}{e^{\cos\theta}}\, d\theta = \frac{1}{e^{\cos v}} - \frac{1}{e} \geqslant 0$$

with $x(-v) = -x(v)$, $y(-v) = y(v)$. The curve $\gamma = (x(v), y(v))$ is symmetric with respect to y-axis with maximum curvature $k = e$ at the origin. Also observe that $y(2n\pi) = 0$ for all $n \in \mathbb{N}$, and

$$\lim_{n\to\infty,\, n\in\mathbb{N}} x(2n\pi) = -\lim_{n\to\infty,\, n\in\mathbb{N}} \int_0^{2n\pi} \sin^2\theta \cdot e^{-\cos\theta}\, d\theta = -\infty,$$

$$\lim_{n\to\infty,\, n\in\mathbb{N}} x(-2n\pi) = +\infty, \qquad x(m\pi) < 0 \quad \text{for all } m \in \mathbb{N}.$$

By $(x'(v), y'(v)) = e^{-\cos v}(\cos v, \sin v)$, together with the positions of the points

$$\big(x(\pi/2), y(\pi/2)\big) = (+, +), \qquad \big(x(\pi), y(\pi)\big) = (-, +),$$
$$\big(x(3\pi/2), y(3\pi/2)\big) = (-, +), \qquad \big(x(2\pi), y(2\pi)\big) = (-, 0)$$

we have a feeling of the picture for $v \in [0, 2\pi]$. We also note that

$$\log k \begin{cases} > 0 & \text{for } v \in [0, \pi/2) \cup (3\pi/2, 2\pi], \\ = 0 & \text{for } v = \pi/2, 3\pi/2, \\ < 0 & \text{for } v \in (\pi/2, 3\pi/2). \end{cases}$$

Hence the portion of the curve on $v \in [0, \pi/2) \cup (3\pi/2, 2\pi]$ is moving in the direction $N(v) = (-\sin v, \cos v)$; while the portion of the curve on $v \in (\pi/2, 3\pi/2)$ is moving in the direction $-N(v)$. As a whole, the curve is translating upward in the direction $V = (0, 1)$. At the two points where $k(v) = 1$, it has no movement in their normal directions. As v goes from 0 to 2π, it is oriented in the counterclockwise direction. See Fig. 1 for the curve $(x(v), y(v))$.

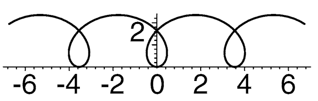

Fig. 1. The curve $(x(v), y(v)) = (\int_0^v \frac{\cos\theta}{e^{\cos\theta}} d\theta, \frac{1}{e^{\cos v}} - \frac{1}{e})$, $v \in [-10, 10]$.

Example 2. Take

$$F(k) = k - 1, \quad F : \mathbb{R}_+ \to (-1, \infty).$$

This flow is interesting in that it is the gradient flow of the functional

$$E(\gamma) = length - area = \int_\gamma ds - \frac{1}{2}\left(\int_\gamma x\,dy - y\,dx\right)$$

with respect to the L^2 inner product $\langle f, g\rangle = \int_\gamma fg\,ds$, where ds is the arc length parameter and γ is a simple closed curve in $\mathbb{R}^2$, see Yagisita [19]. Now $k(v) = 1 + \cos v \geqslant 0$ for all $v \in (-\infty, \infty)$. To avoid $k(v) = 0$, we take $v \in (-\pi, \pi)$. Hence

$$x(v) = \int_0^v \frac{\cos\theta}{1+\cos\theta}\,d\theta = v - \tan\frac{v}{2}, \quad v \in (-\pi, \pi), \quad \text{and}$$

$$y(v) = \int_0^v \frac{\sin\theta}{1+\cos\theta}\,d\theta = \log\left(\frac{2}{1+\cos v}\right) \geqslant 0, \quad v \in (-\pi, \pi).$$

Thus the curve $\gamma = (x(v), y(v))$ is symmetric with respect to y-axis with curvature decreasing to zero at infinity. Figure 2 depicts the curve $(x(v), y(v))$.

Example 3. Consider the most important contracting flow

$$F(k) = k^\alpha, \quad F : \mathbb{R}_+ \to \mathbb{R}_+,\ \alpha > 0 \text{ is a constant.}$$

Now $k(v) = (\cos v)^{1/\alpha}$ and we restrict $v \in (-\pi/2, \pi/2)$ to require $k(v) > 0$. Now

$$x(v) = \int_0^v (\cos\theta)^{1-\frac{1}{\alpha}}\,d\theta, \quad v \in (-\pi/2, \pi/2), \quad \text{and}$$

$$y(v) = \int_0^v \frac{\sin\theta}{(\cos\theta)^{1/\alpha}}\,d\theta = \begin{cases} \frac{\alpha}{\alpha-1}\left[1 - (\cos v)^{1-\frac{1}{\alpha}}\right], & \alpha \neq 1, \\ -\log\cos v, & \alpha = 1, \end{cases} \quad v \in (-\pi/2, \pi/2).$$

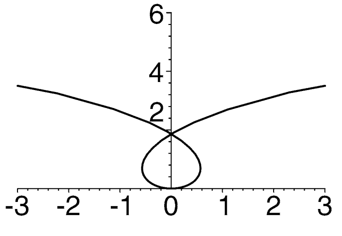

Fig. 2. The curve $(x(v), y(v)) = (v - \tan\frac{v}{2}, \log(\frac{2}{1+\cos v}))$, $v \in (-\pi, \pi)$.

The curve $\gamma = (x(v), y(v))$ is symmetric with respect to y-axis. As $\cos v$ is like $(\pi/2) - v$ when v is near $\pi/2$, we get

$$\lim_{v\to\pi/2} x(v) = \int_0^{\pi/2} (\cos\theta)^{1-\frac{1}{\alpha}}\, d\theta = \begin{cases} \text{finite number } x_* > 0, & \text{when } \alpha > \frac{1}{2}, \\ +\infty, & \text{when } 0 < \alpha \leqslant \frac{1}{2}. \end{cases}$$

Similarly

$$\lim_{v\to\pi/2} y(v) = \begin{cases} \text{finite}, & \text{when } \alpha > 1, \\ +\infty, & \text{when } 0 < \alpha \leqslant 1. \end{cases}$$

If we express the picture as a graph $y = h(x)$, then $h(x)$ is symmetric with respect to y-axis satisfying the ODE

$$h''(x) = \left(1 + \left(h'(x)\right)^2\right)^{\frac{3\alpha-1}{2\alpha}}. \tag{3.11}$$

When $0 < \alpha \leqslant 1/2$, $h(x)$ is defined on $x \in (-\infty, \infty)$. By ODE (3.11), we can obtain the following: for $0 < \alpha < 1/2$, $h(x)$ is asymptotically like $x^{(1-\alpha)/(1-2\alpha)}$ as $x \to \infty$; when $\alpha = 1/2$, $h(x)$ is asymptotically like e^x as $x \to \infty$; when $\alpha > 1/2, h(x)$ is defined only on a finite interval $(-x_*, x_*)$ with $h'(x)$ blowing up at $x = x_*$; for $1/2 < \alpha < 1$, $h(x)$ is asymptotically like $(x_* - x)^{(1-\alpha)/(1-2\alpha)}$ as $x \nearrow x_*$; for $\alpha = 1$, $h(x)$ is given by $h(x) = \log\sec x$ with $x_* = \pi/2$ (grim reaper); for $\alpha > 1$, $h(x)$ approaches a finite limit h_* as $x \nearrow x_*$. Also see the book by Chou and Zhu [13, p. 28]. For various $\alpha > 0$, the curves are shown in Fig. 3.

Example 4. Consider the expanding flow

$$F(k) = -k^{-\alpha}, \quad F : \mathbb{R}_+ \to \mathbb{R}_- = (-\infty, 0),\ \alpha > 0 \text{ is a constant.}$$

We have

$$k(v) = \frac{1}{(-\cos v)^{1/\alpha}} > 0, \quad v \in (\pi/2, 3\pi/2).$$

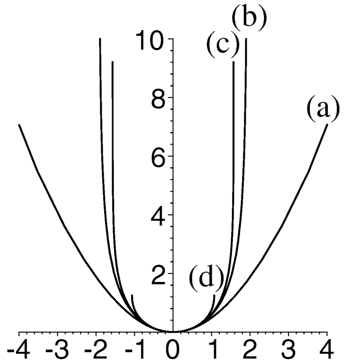

Fig. 3. The curves $(x(v), y(v))$ in Example 3, $v \in (-\pi/2, \pi/2)$, with (a) $\alpha = 0.3$, (b) $\alpha = 0.8$, (c) $\alpha = 1$ (grim reaper), and (d) $\alpha = 5$.

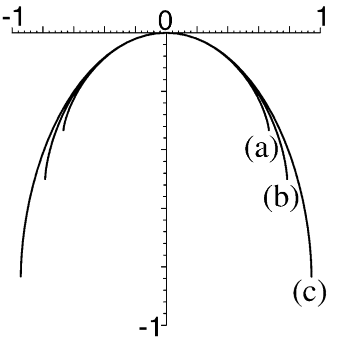

Fig. 4. The curves $(x(v), y(v)) = (\int_\pi^v \cos\theta \cdot (-\cos\theta)^{\frac{1}{\alpha}}\, d\theta, \frac{\alpha}{\alpha+1}[(-\cos v)^{1+\frac{1}{\alpha}} - 1])$, $v \in (\pi/2, 3\pi/2)$, with (a) $\alpha = 0.5$, (b) $\alpha = 1$, and (c) $\alpha = 5$.

Hence if we assume that $(x(\pi), y(\pi)) = (0, 0)$, we would have

$$x(v) = \int_\pi^v \cos\theta \cdot (-\cos\theta)^{\frac{1}{\alpha}}\, d\theta, \quad v \in (\pi/2, 3\pi/2), \quad \text{and}$$

$$y(v) = \int_\pi^v \sin\theta \cdot (-\cos\theta)^{\frac{1}{\alpha}}\, d\theta = \frac{\alpha}{\alpha+1}\big[(-\cos v)^{1+\frac{1}{\alpha}} - 1\big] \leqslant 0, \quad v \in (\pi/2, 3\pi/2),$$

where $x(v)$ is an odd function of v with respect to $v = \pi$ and $y(v)$ is an even function with respect to $v = \pi$. The curve $\gamma = (x(v), y(v))$ is symmetric with respect to y-axis with minimum curvature $k = 1$ occurred at $(x(\pi), y(\pi)) = (0, 0)$. When v approaches $\pi/2$ or $3\pi/2$, both $x(v)$ and $y(v)$ are finite but $k(v)$ approaches ∞. In the special case when $\alpha = 1$, we can integrate $x(v)$ to get

$$x(v) = -\frac{\sin 2v}{4} + \frac{\pi - v}{2}, \qquad y(v) = \frac{\cos 2v}{4} - \frac{1}{4}, \quad v \in (\pi/2, 3\pi/2).$$

By $(x'(v), y'(v)) = -\cos v(\cos v, \sin v)$, $v \in (\pi/2, 3\pi/2)$, as v is increasing γ is traversing from a point in the 4th quadrant to the origin and then to a point in the 3rd quadrant. The normal $N = (-\sin v, \cos v)$ of γ is pointing downward for $v \in (\pi/2, 3\pi/2)$. Since γ is moving in the direction $F(k)N = -k^{-\alpha}N$, $k > 0$, it is moving *upward*. For various $\alpha > 0$, the curves are drawn in Fig. 4.

Note that in the above four examples, the ranges of the speed function $F(z)$ on $z \in (0, \infty)$ are $(-\infty, \infty)$, $(-1, \infty)$, $(0, \infty)$, $(-\infty, 0)$, respectively.

3.2. The speed of a translational sss

Let $A = |V| > 0$ be the speed of translation for a translational sss. Now $F^{-1}(A\cos v) = k(v)$ and if $v = 0$ is in the domain of $(x(v), y(v))$, we have the parametric representation

$$x(v) = \int_0^v \frac{\cos\theta}{F^{-1}(A\cos\theta)}\, d\theta, \qquad y(v) = \int_0^v \frac{\sin\theta}{F^{-1}(A\cos\theta)}\, d\theta. \tag{3.12}$$

We first note that in Lemma 3.5 we assume the translational sss has unit speed. For arbitrary speed $A = |V| > 0$, since we still have

$$x(2\pi) = \int_0^{2\pi} \frac{\cos\theta}{F^{-1}(A\cos\theta)}\, d\theta < 0$$

there does not exist a convex closed embedded curve as a translational sss with speed A for any curvature function $F(k)$.

For any increasing function $F(z):\mathbb{R}_+ \to \mathbb{R}$, one can *always* choose $A > 0$ so that the image of F has nonempty intersection with $[-A, A]$. Let

$$-\infty \leqslant a = \inf_{z\in(0,\infty)} F(z) < \infty \quad \text{and} \quad -\infty < b = \sup_{z\in(0,\infty)} F(z) \leqslant \infty.$$

We have the following obvious result.

Lemma 3.6. *For any smooth increasing function $F(z):\mathbb{R}_+ \to \mathbb{R}$, there exists a convex translational sss for Eq.* (1.1) *with suitable speed $A > 0$. More precisely*:

(1) *If $b < 0$, any $A > -b$ can be the speed of a translational sss.*
(2) *If $b = 0$, any $A > 0$ can be the speed of a translational sss.*
(3) *If $b > 0$ (include $b = \infty$) and $a \leqslant 0$ (include $a = -\infty$), then any $A > 0$ can be the speed of a translational sss.*
(4) *If $b > 0$ (include $b = \infty$) and $a > 0$, then any $A > a$ can be the speed of a translational sss.*

For the ranges of v and k for each possible value A, we can easily derive the following lemmas. Their proofs are omitted. For convenience we set I, J and L to be the following intervals:

$$\begin{aligned}
I &= \bigl(\cos^{-1}(b/A), 2\pi - \cos^{-1}(b/A)\bigr),\\
J &= \bigl(\cos^{-1}(b/A), \cos^{-1}(a/A)\bigr) \cup \bigl(-\cos^{-1}(a/A), -\cos^{-1}(b/A)\bigr) := J_1 \cup J_2,\\
L &= \bigl(-\cos^{-1}(a/A), \cos^{-1}(a/A)\bigr).
\end{aligned}$$

We have the following three cases.

Lemma 3.7. *When $b \leqslant 0$, we have*

values of a, b	*value of A*	*range for v*	*range for k*
$b < 0,\ a = -\infty$	$A > -b$	I	$[F^{-1}(-A), \infty)$
$b < 0,\ -\infty < a < 0$	$-b < A < -a$	I	$[F^{-1}(-A), \infty)$
$b < 0,\ -\infty < a < 0$	$A \geqslant -a$	J	$(0, \infty)$ *on* J_1; $(0, \infty)$ *on* J_2
$b = 0,\ a = -\infty$	$A > 0$	$(\pi/2, 3\pi/2)$	$[F^{-1}(-A), \infty)$
$b = 0,\ -\infty < a < 0$	$0 < A < -a$	$(\pi/2, 3\pi/2)$	$[F^{-1}(-A), \infty)$
$b = 0,\ -\infty < a < 0$	$A \geqslant -a$	J (*with* $b = 0$)	$(0, \infty)$ *on* J_1; $(0, \infty)$ *on* J_2.

Lemma 3.8. *When $b > 0$ is finite, we have*

values of a, b	*value of* A	*range for* v	*range for* k
$b > 0,\ a = -\infty$	$0 < A < b$	$(-\infty, \infty)$	$[F^{-1}(-A), F^{-1}(A)]$
$b > 0,\ a = -\infty$	$A \geqslant b$	I	$[F^{-1}(-A), \infty)$
$b > 0,\ -\infty < a < 0$	$0 < A < \min\{\lvert a\rvert, \lvert b\rvert\}$	$(-\infty, \infty)$	$[F^{-1}(-A), F^{-1}(A)]$
$b > 0,\ b > -a > 0$	$-a \leqslant A < b$	L	$(0, F^{-1}(A)]$
$b > 0,\ b > -a > 0$	$A \geqslant b$	J	$(0, \infty)$ *on* J_1; $(0, \infty)$ *on* J_2
$b = -a > 0$	$A \geqslant b$	J	$(0, \infty)$ *on* J_1; $(0, \infty)$ *on*J_2
$-a > b > 0$	$b \leqslant A < -a$	I	$[F^{-1}(-A), \infty)$
$-a > b > 0$	$A \geqslant -a$	J	$(0, \infty)$ *on* J_1; $(0, \infty)$ *on* J_2
$b > 0,\ a = 0$	$0 < A < b$	$(-\pi/2, \pi/2)$	$(0, F^{-1}(A)]$
$b > 0,\ a = 0$	$A \geqslant b$	J (*with* $a = 0$)	$(0, \infty)$ *on* J_1; $(0, \infty)$ *on* J_2
$b > 0,\ a > 0$	$a < A < b$	L	$(0, F^{-1}(A)]$
$b > 0,\ a > 0$	$A \geqslant b$	J	$(0, \infty)$ *on* J_1; $(0, \infty)$ *on* J_2.

Lemma 3.9. *When $b = \infty$, we have*

values of a, b	*value of* A	*range for* v	*range for* k
$b = \infty,\ a = -\infty$	$A > 0$	$(-\infty, \infty)$	$[F^{-1}(-A), F^{-1}(A)]$
$b = \infty,\ -\infty < a < 0$	$0 < A < -a$	$(-\infty, \infty)$	$[F^{-1}(-A), F^{-1}(A)]$
$b = \infty,\ -\infty < a < 0$	$A \geqslant -a$	L	$(0, F^{-1}(A)]$
$b = \infty,\ a = 0$	$A > 0$	$(-\pi/2, \pi/2)$	$(0, F^{-1}(A)]$
$b = \infty,\ a > 0$	$A > a$	L	$(0, F^{-1}(A)]$.

One particular interest is when $F(z_0) = 0$ for some $z_0 > 0$. For small speed $A > 0$, (3.12) is defined for all $v \in (-\infty, \infty)$ and $F^{-1}(A\cos v)$ is close to z_0. The corresponding translational sss has self-intersections. It lies on the upper half plane, touching the x-axis at the values $v = 0$ and $\pm 2n\pi$, $n \in \mathbb{N}$. When $A \to 0$, the solution converges to the circle passing through $(0, 0)$ with curvature $k = z_0$.This circle does not translate at all as now $F(k)N \equiv F(z_0)N \equiv 0$. This demonstrates that simple closed curve (circle) can be the limit of a family of self-intersecting translational sss. It can also be seen from (3.12) by

$$\big(x(v), y(v)\big) \approx \frac{1}{z_0}\left(\int_0^v \cos\theta\, d\theta, \int_0^v \sin\theta\, d\theta\right) = \frac{1}{z_0}\big(\sin v, (1 - \cos v)\big), \quad v \in (-\infty, \infty), \tag{3.13}$$

when A is small.

To see the effect of A on the curvature of a translational sss, we first observe the following: write $x(v)$ and $y(v)$ in (3.12) as $x(v, A)$ and $y(v, A)$. When $F(k) = k^\alpha$ we have

$$\big(x(v, A), y(v, A)\big) = A^{-\frac{1}{\alpha}}\big(x(v, 1), y(v, 1)\big). \tag{3.14}$$

Similarly for $F(k) = -k^{-\alpha}$ we have

$$\big(x(v,A), y(v,A)\big) = A^{\frac{1}{\alpha}}\big(x(v,1), y(v,1)\big). \tag{3.15}$$

In both cases the shape is changed by a dilation. Thus $k(v,A) = A^{\frac{1}{\alpha}}k(v,1)$ in the first case and $k(v,A) = A^{-\frac{1}{\alpha}}k(v,1)$ in the second case. For other $F(k)$, fixing v we may differentiate with respect to A to get

$$\frac{\partial k(v,A)}{\partial A} = \frac{\partial}{\partial A}F^{-1}(A\cos v) = \frac{\cos v}{F'(k(v,A))} = \frac{F(k(v,A))}{AF'(k(v,A))} \tag{3.16}$$

and use it to see the effect of A on the curvature. For example, when $F(z)$ is positive everywhere, we have $\partial k(v,A)/\partial A > 0$ for all v; when $F(z)$ is negative everywhere, we have $\partial k(v,A)/\partial A < 0$ for all v; and when $F(k) = e^k$, we have $\partial k(v,A)/\partial A = 1/A$ for all v.

3.3. Examples of general-speed translational sss

We give several examples again and see how the speed A changes the shape of a translational sss.

Example 5. For $F(k) = \log k : \mathbb{R}_+ \to \mathbb{R}$, we have $k(v) = e^{A\cos v}$ and the translational sss in the direction (0, 1) with speed $A > 0$ is given parametrically by

$$x(v) = \int_0^v \frac{\cos\theta}{e^{A\cos\theta}}\,d\theta, \quad v \in (-\infty,\infty),\ x(-v) = -x(v), \quad \text{and}$$

$$y(v) = \int_0^v \frac{\sin\theta}{e^{A\cos\theta}}\,d\theta = \frac{1}{A}\left[\frac{1}{e^{A\cos v}} - \frac{1}{e^A}\right] \geqslant 0, \quad v \in (-\infty,\infty). \tag{3.17}$$

We have $y(2n\pi) = 0$ for all $n \in \mathbb{Z}$ and $\lim_{n\to\infty, n\in\mathbb{N}} x(2n\pi) = -\infty$ with $x(2\pi) < 0$. By (3.13), we know the behavior of the translational sss for small $A > 0$. For large A, since

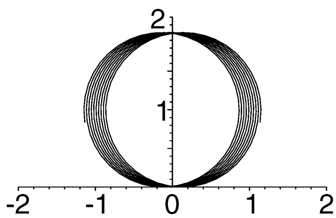

Fig. 5. The curve $(x(v), y(v)) = (\int_0^v \frac{\cos\theta}{e^{A\cos\theta}}\,d\theta, \frac{1}{A}[\frac{1}{e^{A\cos v}} - \frac{1}{e^A}])$, $v \in [-30, 30]$, $A = 0.01$.

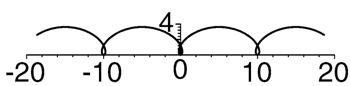

Fig. 6. The curve $(x(v), y(v)) = (\int_0^v \frac{\cos\theta}{e^{A\cos\theta}}\,d\theta, \frac{1}{A}[\frac{1}{e^{A\cos v}} - \frac{1}{e^A}])$, $v \in [-10, 10]$, $A = 2$.

$y(v)$ is increasing on $(0, \pi)$, decreasing on $(\pi, 2\pi)$, $y(v) > 0$ on $(0, 2\pi)$, we have $y(\pi, A) = A^{-1}(e^A - e^{-A}) \to \infty$ as $A \to \infty$ and

$$\lim_{A\to\infty} x(\pi, A) \leqslant \lim_{A\to\infty}\left(-A\int_{\pi/2}^{\pi} \sin^2\theta \cdot e^{-A\cos\theta}\, d\theta\right) = -\infty.$$

Similarly $\lim_{A\to\infty} x(2\pi, A) = -\infty$. Figure 5 is the curve for $A = 0.01$ and Fig. 6 is for $A = 2$.

Example 6. For $F(k) = k - 1 : \mathbb{R}_+ \to (-1, \infty)$, we have $k(v) = 1 + A\cos v$. If $A \in (0, 1)$, then one allow $v \in (-\infty, \infty)$ with

$$x(v) = \int_0^v \frac{\cos\theta}{1 + A\cos\theta}\, d\theta, \quad v \in (-\infty, \infty), \quad \text{and}$$

$$y(v) = \int_0^v \frac{\sin\theta}{1 + A\cos\theta}\, d\theta = \frac{1}{A}\log\left(\frac{1 + A}{1 + A\cos v}\right) \geqslant 0, \quad v \in (-\infty, \infty).$$

For $A > 1$, we must confine the domain to be $v \in I := (-\cos^{-1}(-1/A), \cos^{-1}(-1/A))$ in order to make $k(v) = 1 + A\cos v > 0$. As $A \to +\infty$, the interval I becomes $I_\infty = [-\pi/2, \pi/2]$ and

$$\lim_{A\to\infty} x(\pi/2, A) = \lim_{A\to\infty}\int_0^{\pi/2} \frac{\cos\theta}{1 + A\cos\theta}\, d\theta = \lim_{A\to\infty} y(\pi/2, A) = 0.$$

See Fig. 7 for the curves with $A = 10$ and $A = 100$.

Example 7. Consider $F(k) = e^k : \mathbb{R}_+ \to (1, \infty)$. This flow has no unit-speed translational sss for $A = 1$. For $A > 1$, we can solve $e^k = A\cos v$ to get $k(v) = \log(A\cos v) > 0$ on the interval $v \in$

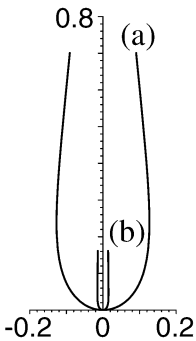

Fig. 7. The curve $((x(v), y(v)) = (\int_0^v \frac{\cos\theta}{1+A\cos\theta}\, d\theta, \frac{1}{A}\log(\frac{1+A}{1+A\cos v}))$, $v \in (-\cos^{-1}(-1/A), \cos^{-1}(-1/A))$, with (a) $A = 10$ and (b) $A = 100$.

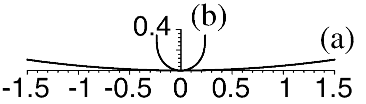

Fig. 8. The curve $(x(v), y(v)) = (\int_0^v \frac{\cos\theta}{\log(A\cos\theta)}\,d\theta, \int_0^v \frac{\sin\theta}{\log(A\cos\theta)}\,d\theta)$, $v \in (-\cos^{-1}(1/A), \cos^{-1}(1/A))$, with (a) $A = 1.1$ and (b) $A = 100$.

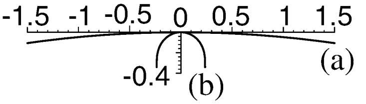

Fig. 9. The curve $(x(v), y(v)) = (\int_\pi^v \frac{\cos\theta}{\log(-A\cos\theta)}\,d\theta, \int_\pi^v \frac{\sin\theta}{\log(-A\cos\theta)}\,d\theta)$, $v \in (\cos^{-1}(-1/A), 2\pi - \cos^{-1}(-1/A))$, with (a) $A = 1.1$ and (b) $A = 100$.

$I := (-\cos^{-1}(1/A), \cos^{-1}(1/A))$. As $A \to 1^+$, the interval $I = (-\delta, \delta)$ is small. As $A \to +\infty$, the interval I becomes $I_\infty = (-\pi/2, \pi/2)$. Now for fixed $v > 0$ in I_∞

$$\lim_{A\to\infty} x(v, A) = \lim_{A\to\infty} \int_0^v \frac{\cos\theta}{\log(A\cos\theta)}\,d\theta = \lim_{A\to\infty} y(v, A) = \lim_{A\to\infty} \int_0^v \frac{\sin\theta}{\log(A\cos\theta)}\,d\theta = 0.$$

See Fig. 8 for the curves with $A = 1.1$ and $A = 100$.

Example 8. Consider $F(k) = -e^{\frac{1}{k}} : \mathbb{R}_+ \to (-\infty, -1)$. This flow has no unit-speed translational sss for $A = 1$. For $A > 1$, we can solve $-e^{\frac{1}{k}} = A\cos v$ to get $k(v) = 1/[\log(-A\cos v)] > 0$ on the interval $v \in I := (\cos^{-1}(-1/A), 2\pi - \cos^{-1}(-1/A))$. As $A \to 1^+$, the interval $I = (\pi - \delta, \pi + \delta)$ is small. As $A \to +\infty$, the interval I becomes $I_\infty = (\pi/2, 3\pi/2)$. Now for fixed $v > \pi$ in I_∞

$$\lim_{A\to\infty} x(v, A) = \lim_{A\to\infty} \int_\pi^v \frac{\cos\theta}{\log(-A\cos\theta)}\,d\theta = \lim_{A\to\infty} y(v, A) = \lim_{A\to\infty} \int_\pi^v \frac{\sin\theta}{\log(-A\cos\theta)}\,d\theta = 0.$$

See Fig. 9 for the curves with $A = 1.1$ and $A = 100$.

Acknowledgments

The authors thank the support of a grant from the National Science Council of Taiwan. We would also like to thank the referee for some valuable comments on the paper. They have been incorporated into Section 1.

References

[1] B. Andrews, Evolving convex curves, Calc. Var. Partial Differential Equations 7 (4) (1998) 315–371.
[2] B. Andrews, Classification of limiting shapes for isotropic curve flows, J. Amer. Math. Soc. 16 (2) (2003) 443–459.
[3] B. Andrews, Contraction of convex hypersurfaces by their affine normal, J. Differential Geom. 43 (1996) 207–230.
[4] U. Abresch, J. Langer, The normalized curve shortening flow and homothetic solutions, J. Differential Geom. 23 (1986) 175–196.
[5] S. Angenent, On the formation of singularities in the curve shortening flow, J. Differential Geom. 33 (1991) 601–633.

[6] S. Angenent, J.L. Velázquez, Asymptotic shape of cusp singularities in curve shortening, Duke Math. J. 77 (1) (1995) 71–110.
[7] B. Chow, Geometric aspects of Aleksandrov reflection and gradient estimates for parabolic equations, Comm. Anal. Geom. 5 (2) (1997) 389–409.
[8] B. Chow, Lecture Notes in Topics in Geometry Course, Univ. of Minnesota, 1992.
[9] B. Chow, D.-H. Tsai, Geometric expansion of convex plane curves, J. Differential Geom. 44 (1996) 312–330.
[10] B. Chow, D.-H. Tsai, Nonhomogeneous Gauss curvature flows, Indiana Univ. Math. J. 47 (3) (1998) 965–994.
[11] K.-S. Chou, G.-X. Li, Optimal systems and invariant solutions for the curve shortening problem, Comm. Anal. Geom. 10 (2) (2002) 241–274.
[12] K.-S. Chou, X.-J. Wang, A logarithmic Gauss curvature flow and the Minkowski problem, Ann. Inst. H. Poincaré Anal. Non Linéaire 17 (6) (2000) 733–751.
[13] K.-S. Chou, X.-P. Zhu, The Curve Shortening Problem, Chapman & Hall/CRC, 2000.
[14] M. Grayson, The heat equation shrinks embedded plane curves to round points, J. Differential Geom. 26 (1987) 285–314.
[15] M. Gage, R. Hamilton, The heat equation shrinking convex plane curves, J. Differential Geom. 23 (1986) 69–96.
[16] R. Hamilton, Isoperimetric estimates for the curve shrinking flow in the plane, in: Ann. of Math. Stud., vol. 137, Princeton Univ. Press, Princeton, 1995, pp. 201–222.
[17] J. Urbas, An expansion of convex hypersurfaces, J. Differential Geom. 33 (1991) 91–125, Correction in: J. Differential Geom. 35 (1992) 763–765.
[18] J. Urbas, Convex curves moving homothetically by negative powers of their curvature, Asian J. Math. 3 (1999) 635–656.
[19] H. Yagisita, Asymptotic behavior of starshaped curves expanding by $V = 1 - k$, Differential Integral Equations 18 (2) (2005) 225–232.